J. Fluid Mech. (1969), *vol.* 36, *part* 4, *pp.* 657–672
Printed in Great Britain

The emission of internal waves by vibrating cylinders

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(Received 25 April 1968)

The paper describes an investigation of the internal waves that are produced in a stratified fluid having constant Brunt–Väisälä frequency by a cylinder which executes small vibrations at a lower frequency. Explicit solutions are found for slender cylinders having arbitrary cross-sections. When the cross-sectional area of the cylinder varies with time it is found necessary in calculating the surface pressures and power output to take account of terms in the governing equations that are significant only at distances from the cylinder comparable to or larger than the scale height of the density variations. For this case a simple expression for the power output is obtained in terms of the rate of change of the cross-sectional area of the cylinder.

When the vibrating cylinder is rigid its cross-sectional area is independent of time and then the expression for the power output is very similar to von Kármán's expression for the drag of a body of revolution in supersonic flow.

In both the above cases it is found that one quarter of the power is radiated in each of the four directions that are inclined at a particular angle to the horizontal.

1. Introduction

Internal waves play an important role in many atmospheric and oceanic geophysical phenomena and in some cases it appears that their production can be regarded as being due to the forced motion of some surface that bounds or partially bounds the fluid in which they occur. An example is provided by the internal waves that can occur in the stably stratified fluid above a strato-cumulus cloud as a result of convection within the cloud (Townsend 1966).

For this reason, and also because it is the standard way in which they are produced in the laboratory, it is of interest to examine the production of internal waves by a vibrating body.

The frequency of vibration of the body must be less than the Brunt–Väisälä frequency of the fluid surrounding it for internal waves to be produced. If it is greater than the acoustic cut-off frequency, which is somewhat greater than the Brunt–Väisälä frequency (Tolstoy 1963), sound waves are emitted and the effects of a small stratification are small in this case. Thus to a good approximation the well-known results for a homogeneous fluid (see, for example, Landau & Lifshitz 1959) may be used. The emission of internal waves is a different problem and has received less attention.

The problem has, however, been considered recently by Mowbray & Rarity (1967) who elucidated many of its main features. The present investigation is complementary to theirs and detailed solutions are obtained for the motions produced by vibrating slender cylinders immersed in a stratified fluid having constant Brunt–Väisälä frequency.

The plan of the paper is as follows: the next section contains the derivation of the equations that govern small motions of a stratified fluid. Terms in these equations that are significant only at distances from the cylinder comparable to or larger than the scale height of the density variations are retained because they have a significant effect on the calculation of the pressures near the cylinder when its cross-sectional area varies with time. Section 3 presents particular solutions of these equations which correspond to a source and a vortex whose strengths vary harmonically with time. Section 4 gives an investigation of the properties of line distributions of sources and vortices and § 5 an account of their use in determining the motions produced by a vibrating slender cylinder.

2. Basic equations

The Euler equations for the two-dimensional motion of a stratified fluid are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{2.1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}-g, \tag{2.2}$$

and the equation of continuity is

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v) = 0. \tag{2.3}$$

Here Oxy is a set of rectangular axes with Ox horizontal and Oy vertically upwards. u, v are the velocity components, ρ is the density, p the pressure and g the acceleration due to gravity.

The condition that the entropy S is constant along a particle path is

$$\frac{\partial S}{\partial t}+u\frac{\partial S}{\partial x}+v\frac{\partial S}{\partial y} = 0,$$

or

$$\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y} = C^2\left\{\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}\right\}, \tag{2.4}$$

where

$$C = \left\{\frac{\partial p}{\partial \rho}\right\}_S^{\frac{1}{2}}$$

is the speed of sound.

Suppose that small motions are taking place in the fluid and put

$$p = p_0+p_1, \quad \rho = \rho_0+\rho_1,$$

where p_0 is the equilibrium pressure and

$$\rho_0 = \rho_0^* \exp(-\beta y), \tag{2.5}$$

where ρ_0^* and β are constants, is the equilibrium density. Then, if the time dependence of each quantity is given by the factor $\exp(-i\omega t)$ the linearized versions of (2.1) to (2.4) are, with subscripts one omitted,

$$-i\omega u+\frac{1}{\rho_0}\frac{\partial p}{\partial x}=0, \tag{2.6}$$

$$-i\omega v+\frac{1}{\rho_0}\frac{\partial p}{\partial y}+\frac{\rho g}{\rho_0}=0, \tag{2.7}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}-\beta v-\frac{i\rho\omega}{\rho_0}=0, \tag{2.8}$$

$$-gv+\beta C^2 v-\frac{i\omega p}{\rho_0}+\frac{i\rho\omega C^2}{\rho_0}=0. \tag{2.9}$$

It is convenient to take p as the fundamental dependent variable and to express the other quantities in terms of it. Hence the set of equations (2.6)–(2.9) is replaced by the equivalent set

$$\frac{\partial^2 p}{\partial y^2}+\beta\frac{\partial p}{\partial y}-\eta^2\frac{\partial^2 p}{\partial x^2}+\frac{\alpha(1-\alpha)\beta^2 p}{1+\eta^2}=0, \tag{2.10}$$

$$u=-\frac{i}{\rho_0\omega}\frac{\partial p}{\partial x}, \tag{2.11}$$

$$v=\frac{i}{\rho_0\omega\eta^2}\left\{\frac{\partial p}{\partial y}+\alpha\beta p\right\}, \tag{2.12}$$

$$\rho=\frac{-1}{g\eta^2}\left\{(1+\eta^2)\frac{\partial p}{\partial y}+\alpha\beta p\right\}, \tag{2.13}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{i\alpha\beta}{\rho_0\omega}\left\{\frac{1}{\eta^2}\frac{\partial p}{\partial y}+\frac{\alpha\beta}{\eta^2}p+\frac{(1-\alpha)\beta p}{1+\eta^2}\right\}. \tag{2.14}$$

In these equations

$$\alpha=\frac{g}{\beta C^2}, \tag{2.15}$$

and
$$\eta^2=\frac{N^2}{\omega^2}-1,$$

where
$$N=\left\{g\beta-\frac{g^2}{C^2}\right\}^{\frac{1}{2}}$$

is the Brunt–Väisälä frequency. C is assumed to be independent of y so that α is a constant that depends only on the physical properties of the medium. α must be less than unity since this is the condition that N be real. If the medium is an isothermal perfect gas

$$C^2=\frac{\gamma p_0}{\rho_0},$$

where γ is the ratio of specific heats, and this in conjunction with the hydrostatic condition gives $\alpha = 1/\gamma$.

If the medium is water, with a variable concentration of salt to produce the stratification, the insertion of typical values into (2.15) shows that α will be small unless $1/\beta$, which is the scale height of the density variations, is greater than about 10 miles. Thus for most laboratory experiments α will be small, but this need not be so for the ocean.

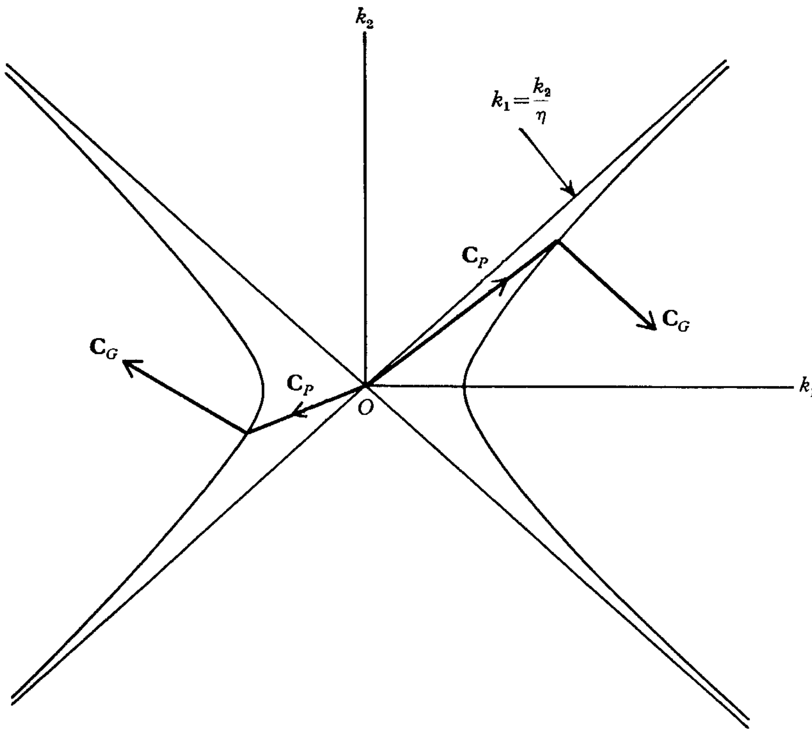

FIGURE 1. Phase and group velocities of internal waves.

The condition that

$$p = \exp(ik_1x + ik_2y - \tfrac{1}{2}\beta y) \tag{2.16}$$

should be a solution of (2.10) is

$$\eta^2k_1^2 - k_2^2 = \beta^2\left\{\frac{1}{4} - \frac{\alpha(1-\alpha)}{1+\eta^2}\right\}. \tag{2.17}$$

If β/k_2 is small, (2.16) is approximately a wave whose phase velocity $\mathbf{C}_P$ is in the direction (k_1, k_2) and whose group velocity $\mathbf{C}_G$ is in the direction of the normal to the hyperbola (2.17) in the sense shown in figure 1. (See, for example, Phillips 1966, p. 175.) Also, if all the vectors $\mathbf{C}_G$ are drawn from the origin O they occupy only the two sectors

$$\left|\frac{y}{x}\right| < \frac{1}{\eta}.$$

3. Solutions for sources and vortices

A particular solution of (2.10) will now be found and it will be shown that the velocity components given in terms of it by (2.11) and (2.12) are those due to a point source.

It may readily be shown that

$$p = \exp(-\tfrac{1}{2}\beta y) f(\zeta),$$

where
$$\zeta^2 = m^2 \left\{\frac{x^2}{\eta^2} - y^2\right\} \quad (3.1)$$

and
$$m = \beta \left\{\frac{1}{4} - \frac{\alpha(1-\alpha)}{1+\eta^2}\right\}^{\frac{1}{2}} \quad (3.2)$$

is a solution of (2.10) provided

$$\frac{d^2 f}{d\zeta^2} + \frac{1}{\zeta}\frac{df}{d\zeta} + f = 0,$$

which is Bessel's equation of zero order. Hence a solution of (2.10) is

$$p_S = \frac{i\eta\rho_0^* \omega}{4} \exp\left(-\frac{\beta y}{2}\right) H_0^{(1)}(\zeta), \quad (3.3)$$

where $H_0^{(1)}(\zeta)$ is the Hankel function of the first kind. Now

$$H_0^{(1)}(\zeta) \sim \left\{\frac{2}{\pi\zeta}\right\}^{\frac{1}{2}} \exp\left\{i\left(\zeta - \frac{\pi}{4}\right)\right\} \quad (\zeta \to \infty),$$

so that
$$p_S \sim \frac{i\eta\rho_0^* \omega}{4} \left\{\frac{2}{\pi\zeta}\right\}^{\frac{1}{2}} \exp\left\{-\frac{\beta y}{2} + i\left(\zeta - \frac{\pi}{4}\right)\right\} \quad (\zeta \to \infty). \quad (3.4)$$

The branch of the two-valued function ζ that is defined by (3.1) will now be chosen so that the Sommerfeld radiation condition is satisfied. It suffices to consider the half-plane $x > 0$ since the pressure field due to a point source at the origin must be an even function of x. It is clear from the discussion at the end of § 2 that at large distances from the origin there will be waves only in the region $|y/x| < 1/\eta$, and that the horizontal component of the phase velocity of the waves in this region will be positive. Hence take

$$\left.\begin{aligned} \zeta &= m\left\{\frac{x^2}{\eta^2} - y^2\right\}^{\frac{1}{2}}, \quad \left|\frac{y}{x}\right| < \frac{1}{\eta} \quad (x > 0) \\ &= im\left\{y^2 - \frac{x^2}{\eta^2}\right\}^{\frac{1}{2}}, \quad \left|\frac{y}{x}\right| \nless \frac{1}{\eta} \quad (x > 0). \end{aligned}\right\} \quad (3.5)$$

For convenience let

$$\left.\begin{aligned} \sigma_+ &= x\sin\mu - y\cos\mu, \\ \sigma_- &= x\sin\mu + y\cos\mu, \end{aligned}\right\} \quad (3.6)$$

where
$$\eta = \cot\mu,$$

so that
$$\zeta = \frac{m}{\cos\mu}\sigma_+^{\frac{1}{2}}\sigma_-^{\frac{1}{2}}. \quad (3.7)$$

Both $\sigma_+^{\frac{1}{2}}$ and $\sigma_-^{\frac{1}{2}}$ have branch points at the origin and the values (3.5) for ζ will be obtained if $\sigma_+^{\frac{1}{2}}$ ($\sigma_-^{\frac{1}{2}}$) is taken to have its principal value for σ_+ (σ_-) real positive and if the indentation around the origin lies in the upper half plane. Now

$$H_0^{(1)}(\zeta) \sim \frac{2i}{\pi}\left(\log\frac{\zeta}{2} + E\right) + 1 + o(1) \quad (\zeta \to 0), \quad (3.8)$$

where E is Euler's number, so that by (3.3) and (3.7)

$$p_S \sim -\frac{\eta\rho_0^*\omega}{2\pi}\left\{\log\left|\frac{m\sigma_+^{\frac{1}{2}}\sigma_-^{\frac{1}{2}}}{2\cos\mu}\right|+E\right\}+\frac{i\eta\rho_0^*\omega}{4}\{H(\sigma_+)+H(\sigma_-)-1\}, \quad \beta r\to 0,\ x>0, \tag{3.9}$$

where H is the Heaviside step function and $r = (x^2+y^2)^{\frac{1}{2}}$ is the distance from the origin.

Now in the region where βr is small the set of equations (2.10) to (2.14) becomes approximately

$$\frac{\partial^2 p}{\partial y^2}-\eta^2\frac{\partial^2 p}{\partial x^2} = 0, \tag{3.10}$$

$$u = \frac{-i}{\rho_0^*\omega}\frac{\partial p}{\partial x}, \tag{3.11}$$

$$v = \frac{i}{\eta^2\rho_0^*\omega}\frac{\partial p}{\partial y}, \tag{3.12}$$

$$\rho = -\frac{1+\eta^2}{\eta^2 g}\frac{\partial p}{\partial y}, \tag{3.13}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \tag{3.14}$$

which are the familiar Boussinesq equations. According to them the velocity components corresponding to p_S as given by (3.9) are

$$u_S = \frac{\cos\mu}{4}\{\delta(\sigma_+)+\delta(\sigma_-)\}+\frac{ix\sin\mu\cos\mu}{2\pi\sigma_+\sigma_-} \quad (x>0), \tag{3.15}$$

$$v_S = \frac{\sin\mu}{4}\{\delta(\sigma_+)-\delta(\sigma_-)\}+\frac{iy\sin\mu\cos\mu}{2\pi\sigma_+\sigma_-} \quad (x>0), \tag{3.16}$$

where δ is the Dirac delta function.

Equations (3.15) and (3.16) hold in $x > 0$. The values of u_S and v_S in $x < 0$ may be obtained by noting that u_S is an odd function of x and v_S an even function.

The velocity given by (3.15) and (3.16) is always in the radial direction so that its value at a point having polar co-ordinates (r, ϕ) can be expressed in the form

$$u_S = (q_{1S}+iq_{2S})\cos\phi, \quad v_S = (q_{1S}+iq_{2S})\sin\phi,$$

where

$$q_{1S} = \tfrac{1}{4}\{\delta(\sigma_+)+\delta(\sigma_-)\}$$

and

$$q_{2S} = \frac{\sin\mu\cos\mu}{2\pi r\sin(\mu-\phi)\sin(\mu+\phi)}.$$

The instantaneous radial velocity is $q_{1S}\cos\omega t + q_{2S}\sin\omega t$ and the main features of the flow are shown in figure 2. It consists of four jets directed outwards along the lines $\sigma_+ = 0$, $\sigma_- = 0$, each having a flux $\frac{1}{4}\cos\omega t$. The out-of-phase speed $q_{2S}\sin\omega t$ is singular along these lines and the total out-of-phase flux from the origin at any instant is zero.

It follows from (3.14) that there is a stream function ψ_S such that

$$u_S = -\frac{\partial\psi_S}{\partial y}, \quad v_S = \frac{\partial\psi_S}{\partial x},$$

and (3.15) and (3.16) give

$$\psi_S = \tfrac{1}{4}\{1+H(\sigma_+)-H(\sigma_-)\}-\frac{i}{4\pi}\log\left|\frac{\sigma_-}{\sigma_+}\right| \quad (x>0)$$
$$= \frac{1}{4}-\frac{i}{4\pi}\log\frac{\sigma_-}{\sigma_+}. \tag{3.17}$$

The constant of integration in (3.17) has been chosen to make ψ_S zero on $x = 0$, $y > 0$. Since ψ_S is an odd function of x this ensures that it is continuous on $x = 0$, $y > 0$.

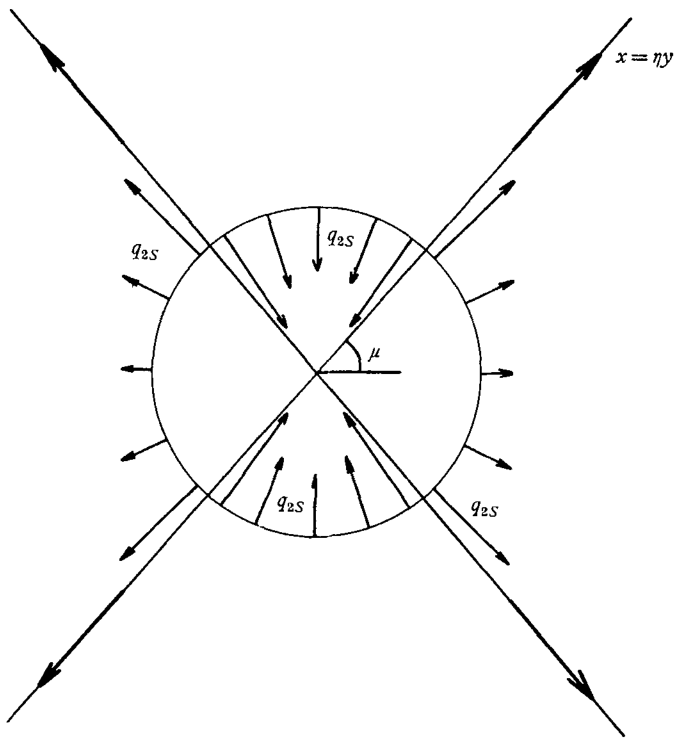

FIGURE 2. Motion due to a source of strength cos ωt in a stratified fluid. The motion is in the radial direction and the in-phase component consists of four jets each having a flux of $\frac{1}{4}$ cos ωt. These are denoted by heavy arrows in the figure. The figure also gives schematic values of q_{2S} where q_{2S} sin ωt is the out-of-phase component.

ψ_S will be called the stream function for a source of unit strength at the origin.

If ψ_S is differentiated with respect to y and then integrated with respect to x the result is the stream function

$$\psi_V = -\frac{i\eta}{4\pi}\log(\sigma_+\sigma_-)$$
$$= \frac{\eta}{4}\{2-H(\sigma_+)-H(\sigma_-)\}-\frac{i\eta}{4\pi}\log|\sigma_+\sigma_-| \quad (x>0). \tag{3.18}$$

ψ_V is defined to be an even function of x and will be called the stream function for a vortex of unit strength at the origin.

4. Fluid velocities due to distributions of sources and vortices

In this section the fluid velocities due to line distributions of sources and vortices will be determined and the results will be used in §5 to determine the motions produced by a vibrating slender cylinder.

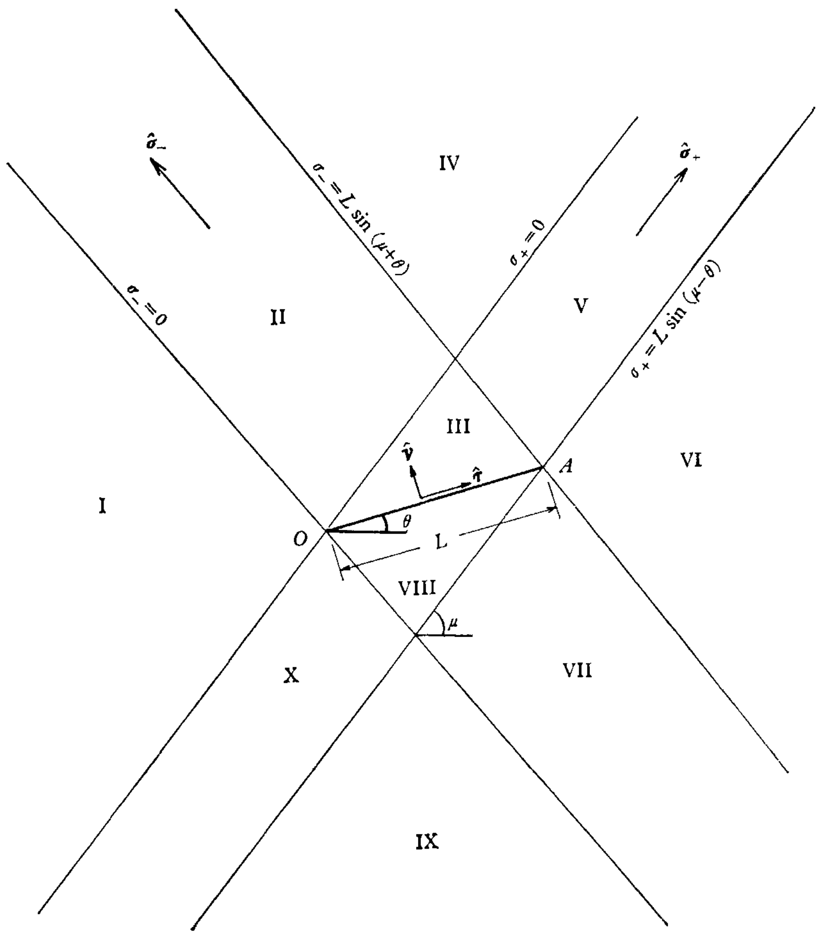

FIGURE 3. Notation for motion due to a distribution of sources or vortices on OA.

A distribution of sources will be considered first.

Let OA be a straight line of length L (see figure 3) that is inclined at an angle θ to Ox and on which there is a distribution of sources of strength $f(t)$ where t is the arc length from O. Then it follows from (3.17) that at distances small compared to $1/\beta$

$$\psi = \int_0^L \{K_1(t) + iK_2(t)\} f(t)\, dt,$$

where $\quad K_1(t) = \tfrac{1}{4}\{1 + H(\sigma_+ - \tau_+(t)) - H(\sigma_- - \tau_-(t)\} \quad (x > t\cos\theta),$

and is an odd function of $x - t\cos\theta$, and

$$K_2(t) = -\frac{1}{4\pi}\log\left|\frac{\sigma_- - \tau_-(t)}{\sigma_+ - \tau_+(t)}\right|.$$

Here σ_+ and σ_- are defined by (3.6) and

$$\tau_+(t) = t\sin(\mu-\theta), \quad \tau_-(t) = t\sin(\mu+\theta) \tag{4.1}$$

are the values of σ_+ and σ_- respectively for $x = t\cos\theta$, $y = t\sin\theta$.

An alternative, and very convenient, way of writing the above expression for ψ is

$$\psi = \frac{i}{4\pi\sin(\mu-\theta)}\int_{\Gamma_+} f_+(\tau_+)\log(\sigma_+-\tau_+)\,d\tau_+ - \frac{i}{4\pi\sin(\mu+\theta)}\int_{\Gamma_-} f_-(\tau_-)\log(\sigma_--\tau_-)\,d\tau_-, \tag{4.2}$$

where $f(t) = f_+(\tau_+) = f_-(\tau_-)$.

The path of integration Γ_+ is the interval $(O, L\sin[\mu-\theta])$ of the real τ_+ axis with an indentation either above or below the point $\tau_+ = \sigma_+$, and Γ_- is the interval $(O, L\sin[\mu+\theta])$ of the real τ_- axis with an indentation at $\tau_- = \sigma_-$. For points that lie above OA and its extensions the indentation is below σ_+ and above σ_- and for points below OA the indentation is above σ_+ and below σ_-.

Region	Second last term	Last term
I, IV, VI, IX	Zero	Zero
II	Zero	+
III	+	+
V	+	Zero
VII	Zero	−
VIII	−	−
X	−	Zero

TABLE 1. Signs of last two terms in equations (4.3) and (5.4).

It follows from (4.2) that the fluid velocity is

$$v = \frac{-i\hat{\sigma}_+}{4\pi\sin(\mu-\theta)}\int_{\Gamma_+}\frac{f_+(\tau_+)}{\tau_+-\sigma_+}d\tau_+ + \frac{i\hat{\sigma}_-}{4\pi\sin(\mu+\theta)}\int_{\Gamma_-}\frac{f_-(\tau_-)}{\tau_--\sigma_-}d\tau_-,$$

where $\hat{\sigma}_+(\hat{\sigma}_-)$ is a unit vector parallel to the lines on which $\sigma_+(\sigma_-)$ is constant in the sense shown in figure 3.

Application of the residue theorem gives

$$v = \frac{-i\hat{\sigma}_+}{4\pi\sin(\mu-\theta)}P\int_0^{L\sin(\mu-\theta)}\frac{f_+(\tau_+)}{\tau_+-\sigma_+}d\tau_+ + \frac{i\hat{\sigma}_-}{4\pi\sin(\mu+\theta)}P \times\int_0^{L\sin(\mu+\theta)}\frac{f_-(\tau_-)}{\tau_--\sigma_-}d\tau_- \pm \frac{f_+(\sigma_+)\hat{\sigma}_+}{4\sin(\mu-\theta)} \pm \frac{f_-(\sigma_-)\hat{\sigma}_-}{4\sin(\mu+\theta)}, \tag{4.3}$$

where P denotes the Cauchy principal value. To specify the signs of the last two terms it is convenient to consider separately the various regions, shown in figure 3, into which the Oxy plane is divided by the lines through O and A on which σ_+ or σ_- is constant. Table 1 gives the signs of the terms and also shows in which regions each is zero.

In the applications the normal and tangential components of the velocity on either side of the line OA will be needed. Let $\hat{\nu}$ and $\hat{\tau}$ be unit vectors in the direction of the normal and tangent to OA in the senses shown in figure 3 and let $\mathbf{v}_U$ and $\mathbf{v}_L$ denote the velocities on the upper and lower sides of the line respectively. Then it follows from (4.3) that

$$\mathbf{v}_U = \frac{f(s)\sin 2\theta\hat{\mathbf{r}}}{4\sin(\mu-\theta)\sin(\mu+\theta)} - \frac{i\sin 2\mu\hat{\mathbf{r}}}{4\pi\sin(\mu-\theta)\sin(\mu+\theta)}\int_0^L \frac{f(t)}{t-s}\,dt + \frac{f(s)\hat{\mathbf{p}}}{2}, \quad (4.4)$$

$$\mathbf{v}_L = -\frac{f(s)\sin 2\theta\hat{\mathbf{r}}}{4\sin(\mu-\theta)\sin(\mu+\theta)} - \frac{i\sin 2\mu\hat{\mathbf{r}}}{4\pi\sin(\mu-\theta)\sin(\mu+\theta)}\int_0^L \frac{f(t)}{t-s}\,dt - \frac{f(s)\hat{\mathbf{p}}}{2}, \quad (4.5)$$

where $x = s\cos\theta$, $y = s\sin\theta$.

For a distribution of vortices of strength $g(t)$ it follows from (3.18) that

$$\psi = \frac{-i\eta}{4\pi\sin(\mu-\theta)}\int_{\Gamma_+} g_+(\tau_+)\log(\sigma_+-\tau_+)\,d\tau_+ - \frac{i\eta}{4\pi\sin(\mu+\theta)}\int_{\Gamma_-} g_-(\tau_-)\log(\sigma_--\tau_-)\,d\tau_-, \quad (4.6)$$

where Γ_+ and Γ_- have the same meaning as in (4.2) and

$$g(t) = g_+(\tau_+) = g_-(\tau_-).$$

Hence

$$\mathbf{v} = \frac{i\eta\hat{\sigma}_+}{4\pi\sin(\mu-\theta)}\int_{\Gamma_+}\frac{g_+(\tau_+)}{\tau_+-\sigma_+}\,d\tau_+ + \frac{i\eta\hat{\sigma}_-}{4\pi\sin(\mu+\theta)}\int_{\Gamma_-}\frac{g_-(\tau_-)}{\tau_--\sigma_-}\,d\tau_-, \quad (4.7)$$

and

$$\mathbf{v}_U = -\frac{\eta\sin 2\mu g(s)\,\hat{\mathbf{r}}}{4\sin(\mu-\theta)\sin(\mu+\theta)} + \frac{i\eta\sin 2\theta\hat{\mathbf{r}}}{4\pi\sin(\mu-\theta)\sin(\mu+\theta)}\int_0^L\frac{g(t)}{t-s}\,dt + \frac{i\eta\hat{\mathbf{p}}}{2\pi}\int_0^L\frac{g(t)}{t-s}\,dt, \quad (4.8)$$

$$\mathbf{v}_L = \frac{\eta\sin 2\mu g(s)\,\hat{\mathbf{r}}}{4\sin(\mu-\theta)\sin(\mu+\theta)} + \frac{i\eta\sin 2\theta\hat{\mathbf{r}}}{4\pi\sin(\mu-\theta)\sin(\mu+\theta)}\int_0^L\frac{g(t)}{t-s}\,dt + \frac{i\eta\hat{\mathbf{p}}}{2\pi}\int_0^L\frac{g(t)}{t-s}\,dt. \quad (4.9)$$

5. Vibrating slender cylinders

Consider an infinite body of stratified fluid whose equilibrium density is given by (2.5). We consider the two-dimensional motion produced in the fluid by the vibration of a horizontal cylinder. Suppose that the generators of the cylinder are normal to the Oxy plane and let osn be a second set of axes in this plane with os inclined at an angle θ to Ox.

The cylinder and its motion are supposed to be such that at all times the various points of its cross-section are close to the segment $o < s < L$ of the os axis. Thus the equations of its upper and lower surfaces can be written

$$\left.\begin{aligned} n &= \epsilon\{H_U(s)+h_U(s)\exp(-i\omega t)\} \\ n &= \epsilon\{H_L(s)+h_L(s)\exp(-i\omega t)\}\end{aligned}\right\} \quad (o < s < L),$$

respectively. Here ϵ is a small parameter and ω is the angular frequency of the vibration.

The boundary condition at the surface of the cylinder is approximately equivalent to the conditions

$$\left.\begin{aligned} q_n^+(s) &= -i\omega\epsilon h_U(s) \\ q_n^-(s) &= -i\omega\epsilon h_L(s)\end{aligned}\right\} \quad (o < s < L),$$

where $q_n^+(s)\exp(-i\omega t)$ $(q_n^-(s)\exp(-i\omega t))$ is the limiting value of the velocity component q_n as the segment $o < s < L$ is approached from above (below).

As in aerofoil theory, it is convenient to consider separately the symmetric and the anti-symmetric problems. Let

$$b(s) = \tfrac{1}{2}\{q_n^+(s) - q_n^-(s)\}$$

and

$$c(s) = \tfrac{1}{2}\{q_n^+(s) + q_n^-(s)\}.$$

In the symmetric problem the boundary condition is

$$\left.\begin{aligned} q_n &= b(s), & n &= 0+, \\ &= -b(s), & n &= 0-, \end{aligned}\right\} \quad (o < s < L) \tag{5.1}$$

and in the anti-symmetric problem it is

$$q_n = c(s), \quad n = 0+ \quad \text{and} \quad 0- \quad (o < s < L). \tag{5.2}$$

5.1 *The symmetric problem*

Equations (4.4) and (4.5) show that the boundary condition (5.1) is satisfied by a distribution of sources on $n = 0, o < s < L$ of strength

$$f(s) = 2b(s). \tag{5.3}$$

It follows from (4.3) and (5.3) that

$$\mathbf{v} = -\frac{i\hat{\sigma}_+}{2\pi\sin(\mu-\theta)} P\int_0^{L\sin(\mu-\theta)} \frac{b_+(\tau_+)}{\tau_+ - \sigma_+}\,d\tau_+ + \frac{i\hat{\sigma}_-}{2\pi\sin(\mu+\theta)} P \times \int_0^{L\sin(\mu+\theta)} \frac{b_-(\tau_-)}{\tau_- - \sigma_-}\,d\tau_- \pm \frac{b_+(\sigma_+)\hat{\sigma}_+}{2\sin(\mu-\theta)} \pm \frac{b_-(\sigma_-)\hat{\sigma}_-}{2\pi\sin(\mu+\theta)}, \tag{5.4}$$

where

$$b(t) = b_+(\tau_+) = b_-(\tau_-),$$

and table 1 refers to this equation as well as equation (4.3). Also from (3.1) and (3.3)

$$p = \frac{i\eta\rho_0^*\omega}{2}\int_0^L \exp\left\{-\frac{\beta}{2}(y - t\sin\theta)\right\} H_0^{(1)}\left\{m\left[\frac{(x-t\cos\theta)^2}{\eta^2} - (y-t\sin\theta)^2\right]^{\frac{1}{2}}\right\} b(t)\,dt, \tag{5.5}$$

and if βr is small (3.8) and (5.5) show that

$$p = -\frac{\eta\rho_0^*\omega Q}{2\pi}\left\{\log(\beta L) + \log\left\{\frac{1}{2\cos\mu}\left[\frac{1}{4} - \frac{\alpha(1-\alpha)}{1+\eta^2}\right]^{\frac{1}{2}}\right\} + E + 1\right\} - \frac{\eta\rho_0^*\omega}{2\pi}\int_0^L \log\left\{\frac{[\sigma_+ - \tau_+(t)][\sigma_- - \tau_-(t)]}{L^2}\right\} b(t)\,dt \tag{5.6}$$

approximately. Here

$$Q = 2\int_0^L b(t)\,dt$$

is the total source strength and may, without loss of generality, be taken to be real.

Suppose that Q is not zero. Then to a first approximation p is constant throughout the region where βr is small:

$$p = -\frac{\eta\rho_0^*\omega Q}{2\pi}\log(\beta L). \tag{5.7}$$

Now the time average of the power radiated across a curve of length l is

$$\frac{1}{4}\int_0^l (pq_n^* + p^* q_n)\, ds, \tag{5.8}$$

where s is arc length, q_n the normal velocity and an asterisk denotes the complex conjugate. Applying this result at the surface of the cylinder gives that the time average of the power radiated is approximately

$$\overline{P} = -\frac{\eta\rho_0^*\omega Q^2}{4\pi}\log(\beta L). \tag{5.9}$$

Equation (5.4) shows that there is an in-phase volume flow of $Q/4$ in each of the four directions that makes angles $\pm\mu$, $\pm(\pi-\mu)$ with the horizontal and it follows that one quarter of the above power is radiated in each of these directions.

We now consider the case when Q is zero. (5.6) gives that to a first approximation

$$\begin{aligned} p &= -\frac{\eta\rho_0^*\omega}{2\pi}\int_0^L \log\{[\sigma_+ - \tau_+(t)][\sigma_- - \tau_-(t)]\}\, b(t)\, dt \\ &= -\frac{\eta\rho_0^*\omega}{2\pi\sin(\mu-\theta)}\int_{\Gamma_+} \log(\sigma_+ - \tau_+)\, b_+(\tau_+)\, d\tau_+ \\ &\qquad -\frac{\eta\rho_0^*\omega}{2\pi\sin(\mu+\theta)}\int_{\Gamma_-} \log(\sigma_- - \tau_-)\, b_-(\tau_-)\, d\tau_-. \end{aligned} \tag{5.10}$$

Let r, ϕ be the polar co-ordinates of a point. Then since Q vanishes

$$\int_0^{L\sin(\mu-\theta)} b_+(\tau_+)\, d\tau_+ = \int_0^{L\sin(\mu+\theta)} b_-(\tau_-)\, d\tau_- = 0,$$

and it follows from (5.4) and (5.10) that†

$$\left.\begin{aligned} \mathbf{v} &= O\left(\frac{1}{r^2}\right) \\ p &= O\left(\frac{1}{r}\right) \end{aligned}\right\} \quad (r \to \infty),$$

except for those values of ϕ for which the integrands in (5.4) and (5.10) are singular, i.e. $\phi = \pm\mu$, $\pm(\pi-\mu)$. Hence the cylinder radiates power only in these directions and it may also be shown that the power radiated in the direction μ, for example, is confined to the region V of figure 3. For points in this region

$$\begin{aligned} p = -\frac{\eta\rho_0^*\omega}{2\pi\sin(\mu-\theta)}\Bigg\{&\int_0^{L\sin(\mu-\theta)} \log|\sigma_+ - \tau_+|\, b_+(\tau_+)\, d\tau_+ \\ &+ i\pi\int_{\sigma_+}^{L\sin(\mu-\theta)} b_+(\tau_+)\, d\tau_+\Bigg\} + O\left(\frac{1}{r}\right) \quad (r\to\infty), \end{aligned} \tag{5.11}$$

and

$$v = q\hat{\sigma}_+ + O\left(\frac{1}{r^2}\right) \quad (r \to \infty),$$

† These estimates give the behaviour for r large of the solution that is valid for βr small. The true behaviours for r large, which we do not need, could be derived from (3.4).

where
$$q = \frac{-i}{2\pi \sin(\mu-\theta)} P \int_0^{L\sin(\mu-\theta)} \frac{b_+(\tau_+)}{\tau_+ - \sigma_+}\, d\tau_+ + \frac{b_+(\sigma_+)}{2\sin(\mu-\theta)}. \tag{5.12}$$

For the periodic motion of a rigid cylinder the normal velocities at all points of the surface of the cylinder are in phase so that in this case $b(t)$ can be taken to be real. Denote the time average of the power radiated in the direction μ by $\overline{P}_S/4$. Then (5.8), (5.11) and (5.12) give

$$\frac{\overline{P}_S}{4} = -\frac{\eta\rho_0^*\omega}{4\pi}\int_0^L\int_0^L b(t)\,b(s)\log|s-t|\,dt\,ds.$$

The power radiated in each of the directions $-\mu$, $\pm(\pi-\mu)$ is also found to be $\overline{P}_S/4$.

5.2 *The anti-symmetric problem*

The boundary condition is (5.2) and it follows from (4.8) and (4.9) that it will be satisfied by a distribution of vortices of strength $g(s)$ provided

$$\frac{i\eta}{2\pi}\int_0^L \frac{g(t)}{t-s}\,dt = c(s) \quad (o < s < L). \tag{5.13}$$

The general solution of this integral equation is (see, for example, Carrier, Krook & Pearson 1966, p. 424)

$$g(s) = \frac{2i}{\pi\eta}\left\{\frac{L-s}{s}\right\}^{\frac{1}{2}}\int_0^L \frac{c(t)\,t^{\frac{1}{2}}}{(L-t)^{\frac{1}{2}}(t-s)}\,dt + \frac{k}{s^{\frac{1}{2}}(L-s)^{\frac{1}{2}}}, \tag{5.14}$$

where k is an arbitrary constant, which may be determined as follows:

In the relation (Lamb 1932, p. 204)

$$\frac{D}{Dt}\int_{\mathscr{C}} \mathbf{v}\,.\,d\mathbf{s} = -\int_{\mathscr{C}}\frac{dp}{\rho}, \tag{5.15}$$

take $\mathscr{C}$ to consist of the upper and lower sides of the line OA. Here p is the total and not the perturbation pressure so that, using the hydrostatic condition, the linear approximation to (5.15) is

$$i\omega\int_0^L (\mathbf{v}_U - \mathbf{v}_L)\,.\,\boldsymbol{\tau}\,ds = \frac{g\sin\theta}{\rho_0^*}\int_0^L (\rho_U - \rho_L)\,ds, \tag{5.16}$$

where ρ_U and ρ_L denote the density perturbations on the upper and lower sides of the line OA. Also from (3.12) and (3.13)

$$\frac{\rho}{\rho_0^*} = \frac{i(1+\eta^2)\,\omega v}{g},$$

which in conjunction with (4.8), (4.9) and (5.16) gives

$$(\sin^2\mu - \sin^2\theta)\int_0^L g(t)\,dt = 0.$$

Hence unless $\theta = \pm\mu$, $\pm(\pi-\mu)$ the constant k in (5.14) is determined by the condition

$$\int_0^L g(t)\,dt = 0. \tag{5.17}$$

$\mathbf{v}$ is given in terms of $g(s)$ by (4.7).

Now if (3.11) and (3.12) are expressed in terms of σ_+, σ_- and ψ it may be shown that

$$\frac{\partial p}{\partial \sigma_+} = i\eta\rho_0^*\omega\frac{\partial \psi}{\partial \sigma_+}$$

and

$$\frac{\partial p}{\partial \sigma_-} = -i\eta\rho_0^*\omega\frac{\partial \psi}{\partial \sigma_-}.$$

Hence from (4.6)

$$p = \frac{\eta^2\rho_0^*\omega}{4\pi\sin(\mu-\theta)}\int_{\Gamma_+} g_+(\tau_+)\log(\sigma_+-\tau_+)\,d\tau_+ - \frac{\eta^2\rho_0^*\omega}{4\pi\sin(\mu+\theta)}\int_{\Gamma_-} g_-(\tau_-)\log(\sigma_--\tau_-)\,d\tau_-.$$

For the anti-symmetric motion of a rigid body $c(s)$ can be taken to be real and then, by (5.13), $g(s)$ will be pure imaginary:

$$g(s) = ig_2(s),$$

say.

Then proceeding as in the symmetric problem it can be shown that the time average of the power radiated in each of the directions $\pm\mu$, $\pm(\pi-\mu)$ is

$$\frac{\overline{P}_A}{4} = -\frac{\eta^3\rho_0^*\omega}{16\pi}\int_0^L\int_0^L g_2(t)g_2(s)\log|s-t|\,dt\,ds. \tag{5.18}$$

6. Examples

Details of the solution of a typical symmetric and anti-symmetric problem are given below for illustrative purposes.

6.1. *Symmetric motion of inclined flat plate*

For this case $q_n^+ = U$, $q_n^- = -U$, $b(s) = U$ and $Q = 2LU$. Hence (5.9) gives

$$\overline{P} = -\frac{\eta\rho_0^*\omega L^2U^2}{\pi}\log(\beta L).$$

Also from (5.4)

$$\mathrm{v} = \frac{iU\hat{\sigma}_+}{2\pi\sin(\mu-\theta)}\log\left|\frac{\sigma_+}{L_+-\sigma_+}\right| - \frac{iU\hat{\sigma}_-}{2\pi\sin(\mu+\theta)}\log\left|\frac{\sigma_-}{L_--\sigma_-}\right| \pm\frac{U\hat{\sigma}_+}{2\sin(\mu-\theta)}\pm\frac{U\hat{\sigma}_-}{2\sin(\mu+\theta)},$$

where $L_+ = L\sin(\mu-\theta)$, $L_- = \sin(\mu+\theta)$ and σ_+ and σ_- are defined by (3.6).

Table 1 gives the signs to be taken for the last two terms and also shows the regions in which each is zero. The main features of the flow are apparent from this expression.

6.2. *Anti-symmetric motion of inclined flat plate*

For this case $q_n^+ = U$, $q_n^- = U$ and $c(s) = U$. Hence by (5.14) and (5.17)

$$g(s) = \frac{iU(L-2s)}{\eta s^{\frac{1}{2}}(L-s)^{\frac{1}{2}}}.$$

Equation (5.18) now gives

$$\overline{P}_A = \frac{\pi\eta\rho_0^*\omega L^2 U^2}{8}.$$

By (4.7) the velocities in the various regions of figure 3 are as follows:

$$\mathbf{v} = \frac{U\hat{\sigma}_+}{2\sin(\mu-\theta)}\left\{1 - \frac{|L_+ - 2\sigma_+|}{2[\sigma_+(\sigma_+ - L_+)]^{\frac{1}{2}}}\right\} + \frac{U\hat{\sigma}_-}{2\sin(\mu+\theta)}\left\{1 - \frac{|L_- - 2\sigma_-|}{2[\sigma_-(\sigma_- - L_-)]^{\frac{1}{2}}}\right\}$$

in regions I, IV, VI and IX,

$$= \frac{U\hat{\sigma}_+}{2\sin(\mu-\theta)}\left\{1 - \frac{|L_+ - 2\sigma_+|}{2[\sigma_+(\sigma_+ - L_+)]^{\frac{1}{2}}}\right\} + \frac{U\hat{\sigma}_-}{2\sin(\mu+\theta)} \pm \frac{iU(L_- - 2\sigma_-)\hat{\sigma}_-}{4\sin(\mu+\theta)\sigma_-^{\frac{1}{2}}(L_- - \sigma_-)^{\frac{1}{2}}}$$

in region $\genfrac{}{}{0pt}{}{\text{II}}{\text{VII}}$,

$$= \frac{U\hat{\sigma}_+}{2\sin(\mu-\theta)} + \frac{U\hat{\sigma}_-}{2\sin(\mu+\theta)} \mp \frac{iU(L_+ - 2\sigma_+)\hat{\sigma}_+}{4\sin(\mu-\theta)\sigma_+^{\frac{1}{2}}(L_+ - \sigma_+)^{\frac{1}{2}}}$$

$$\pm \frac{iU(L_- - 2\sigma_-)\hat{\sigma}_-}{4\sin(\mu+\theta)\sigma_-^{\frac{1}{2}}(L_- - \sigma_-)^{\frac{1}{2}}}$$

in region $\genfrac{}{}{0pt}{}{\text{III}}{\text{VIII}}$,

$$= \frac{U\hat{\sigma}_+}{2\sin(\mu-\theta)} + \frac{U\hat{\sigma}_-}{2\sin(\mu+\theta)}\left\{1 - \frac{|L_- - 2\sigma_-|}{2[\sigma_-(\sigma_- - L_-)]^{\frac{1}{2}}}\right\} \mp \frac{iU(L_+ - 2\sigma_+)\hat{\sigma}_+}{4\sin(\mu-\theta)\sigma_+^{\frac{1}{2}}(L_+ - \sigma_+)^{\frac{1}{2}}}$$

in region $\genfrac{}{}{0pt}{}{\text{V}}{\text{X}}$,

where each square root denotes a positive quantity.

The main features of the flow, including the behaviour at large distances from the plate, are apparent from the above expressions.

7. Discussion

The results obtained when the cross-sectional area of the cylinder varies with time exhibit some surprising features. If βL is small, the pressure is approximately uniform throughout the region where βr is small. Also the time average of the power output is

$$\overline{P} = -\frac{\eta\rho_0^*\omega Q^2}{4\pi}\log(\beta L),$$

and this becomes indefinitely large as $\beta L \to 0$ with Q fixed.

For a rigid cylinder Q is zero and the power output is

$$\overline{P}_S = -\frac{\eta\rho_0^*\omega}{\pi}\int_0^L\int_0^L b(t)\,b(s)\log|s-t|\,dt\,ds$$

for the symmetric case and

$$\overline{P}_A = -\frac{\eta^3\rho_0^*\omega}{4\pi}\int_0^L\int_0^L g_2(t)\,g_2(s)\log|s-t|\,dt\,ds$$

for the anti-symmetric case.

Expressions which are applicable in both cases can be found as follows. Let B and C be corresponding points on either side of the chord OA of the cylinder and let OM and OM' be the two lines that are inclined at an angle μ to the vertical, OM being in the first quadrant and OM' in the second. Also let $|\Delta v_\mu(s)|$

($|\Delta v'_\mu(s)|$) be the absolute value of the difference in the velocity components in the direction OM (OM') at B and C. Then using (4.4), (4.5), (4.8), (4.9) and (5.3) to express $b(s)$ and $g_2(s)$ in terms of $|\Delta v_\mu(s)|$ or $|\Delta v'_\mu(s)|$ it is found that in both cases the mean power output is

$$\begin{aligned}\overline{P} &= -\frac{\rho_0^* \omega \sin^2(\mu-\theta)}{4\pi \sin^3\mu \cos\mu} \int_0^L \int_0^L |\Delta v_\mu(t)|\,|\Delta v_\mu(s)| \log|s-t|\,dt\,ds \\ &= -\frac{\rho_0^* \omega \sin^2(\mu+\theta)}{4\pi \sin^3\mu \cos\mu} \int_0^L \int_0^L |\Delta v'_\mu(t)|\,|\Delta v'_\mu(s)| \log|s-t|\,dt\,ds.\end{aligned}$$

These expressions are very similar to von Kármán's expression (see, for example, Sears 1954, p. 224)

$$D = -\frac{\rho_\infty U^2}{4\pi} \int_0^L \int_0^L S''(x) S''(y) \log|y-x|\,dx\,dy$$

for the drag of a body of revolution in supersonic flow. Here $S(x)$ is the cross-sectional area of the body, L its length, U the free stream speed, ρ_∞ the free stream density and dashes denote differentiation.

Acknowledgment is gratefully given to Professor J. J. Mahony for helpful discussions concerning a problem related to the one considered herein. This work was supported by the National Science Foundation under Grant No. GA 641X.

REFERENCES

Carrier, G. F., Krook, M. & Pearson, C. E. 1966 *Functions of a Complex Variable.* New York: McGraw-Hill.

Lamb, H. 1932 *Hydrodynamics.* Cambridge University Press.

Landau, L. D. & Lifshitz, E. M. 1959 *Fluid Mechanics.* London: Pergamon.

Mowbray, D. E. & Rarity, B. S. H. 1967 *J. Fluid Mech.* **28**, 1.

Phillips, O. M. 1966 *The Dynamics of the Upper Ocean.* Cambridge University Press.

Sears, W. R. (ed.) 1954 *General Theory of High Speed Aerodynamics.* Princeton University Press.

Tolstoy, I. 1963 *Rev. Mod. Phys.* **35**, 216.

Townsend, A. A. 1966 *J. Fluid Mech.* **24**, 307.